

GAUGING OF LORENTZ GROUP WZW MODEL BY ITS NULL SUBGROUP

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Abstract

We consider the standard vector gauging of Lorentz group $SO(3, 1)$ WZW model by its non-semisimple null Euclidean subgroup in two dimensions $E(2)$. The resultant effective action of the theory is seen to describe a one dimensional bosonic field in the presence of external charge that we interpret it as a Liouville field. Gauging a boosted $SO(3)$ subgroup, we find that in the limit of the large boost, the theory can be interpreted as an interacting Toda theory. We also take the generalized non-standard bilinear form for $SO(3, 1)$ and gauge both $SO(3)$ and $E(2)$ subgroups and discuss the resultant theories.

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1 Introduction

The appearance of black hole geometry in the context of gauged WZW models was first discovered in [1] and has lead to an extensive study of gauged WZW for different groups and subgroups [2, 3]. In particular in reference [3] the standard vector gauging of the Lorentz group WZW model by its subgroups $SO(3)$ and $SO(2, 1)$ was considered and their singularity structure was studied. These two different subgroups belong to two distinct conjugacy classes of the subgroups of Lorentz group. However, there is another conjugacy class of subgroup, that is the subgroup which is isomorphic to the Euclidean group in two dimensions; $E(2)$. In a previous articles [4, 5] on $SL(2, R)/U(1)$ black hole, it was discovered that taking the corresponding Euclidean subgroup, result an unexpected reduction of the quotient theory and yield a single Liouville field. In [6], we studied this reduction of degrees of freedom and found out that it was the consequence of a large symmetry .

In this article, we consider the target space of Lorentz group WZW model gauged by its vector non-semisimple null subgroup $E(2)$. The resultant target space and dilaton field will be one dimensional again, but when we boost the $SO(3)$ subgroup which tends to $E(2)$ in the limit of infinite boost, there appears an interacting Toda structure. In section two, we consider $SO(3, 1)/SO(3)$ in a suitable gauge, and find the target space metric and the dilaton field which although turn out to be rather complicated. In section three, $SO(3, 1)/E(2)$ is presented, and is found that it describes a one dimensional Liouville theory. In section four, $SO(3, 1)/SO(3)_b$ is discussed, where $SO(3)_b$ is the boosted version of $SO(3)$ group. We will show that in the limit $b \rightarrow \infty$, as expected, the results are the same as the vector gauged $SO(3, 1)/E(2)$ model, and for finite but large boost parameter, an interacting Toda theory appears. In section five, we investigate the generalized non-standard bilinear form of Lorentz group in the construction of gauged models.

Finally, in the appendix we fill out the details of our gauge fixing.

2 VECTOR GAUGING OF $SO(3,1)/SO(3)$ WZW MODEL

In this section we consider the $SO(3,1)$ WZW model vector gauged by its $SO(3)$ subgroup. We call the six generators of $SO(3,1)$ as $J_1, J_2, J_3, K_1, K_2, K_3$. The gauged action can be written as

$$S(g, A, \bar{A}) = S(g) + S_{gauge}(g) = S(g) + \frac{k}{2\pi} \int_{\Sigma} d^2z \text{Tr}(-\bar{A}J + A\bar{J} + A\bar{A} - g\bar{A}g^{-1}A), \quad (2.1)$$

where the gauge fields A, \bar{A} takes their values in $so(3)$ algebra

$$(A, \bar{A}) = (A_1, \bar{A}_1) J_1 + (A_2, \bar{A}_2) J_2 + (A_3, \bar{A}_3) J_3. \quad (2.2)$$

$S(g)$ is the ungauged $SO(3,1)$ WZW model and J, \bar{J} are the corresponding Kac-Moody currents

$$S(g) = \frac{k}{4\pi} \int_{\Sigma} d^2z \text{Tr}(g^{-1}\partial g g^{-1}\bar{\partial} g) - \frac{k}{12\pi} \int_M \text{Tr}(g^{-1}dg)^3, \quad (2.3)$$

$$J = g^{-1}\partial g, \bar{J} = \bar{\partial} g g^{-1}. \quad (2.4)$$

This action is invariant under following vector gauge transformation

$$g \rightarrow h^{-1}g h, A \rightarrow h^{-1}(A - \partial)h, \bar{A} \rightarrow h^{-1}(\bar{A} - \bar{\partial})h, \quad (2.5)$$

$$g \in SO(3,1), h \in SO(3).$$

By using the homomorphism between $SL(2, C)$ and $SO(3,1)$ groups, we fix the gauge and obtain (the details will be discussed in the appendix)

$$g = \begin{pmatrix} r^2 + \frac{1+r^4-2r^2\cos(2\theta)}{2t^2} + \frac{t^2}{2} & r\cos(\theta)\left(\frac{-1+r^2+t^2}{t}\right) & -r\sin(\theta)\left(\frac{-1+r^2+t^2}{t}\right) & \frac{1+r^4-2r^2\cos(2\theta)}{2t^2} - \frac{t^2}{2} \\ r\cos(\theta)\left(\frac{-1+r^2+t^2}{t}\right) & -1 + 2r^2\cos^2(\theta) & -r^2\sin(2\theta) & r\cos(\theta)\left(\frac{-1+r^2+t^2}{t}\right) \\ -r\sin(\theta)\left(\frac{1+r^2+t^2}{t}\right) & -r^2\sin(2\theta) & 1 + 2r^2\sin^2(\theta) & r\sin(\theta)\left(-\frac{1+r^2+t^2}{t}\right) \\ -\frac{1+r^4-2r^2\cos^2(\theta)}{2t^2} + \frac{t^2}{2} & r\cos(\theta)\left(\frac{1-r^2+t^2}{t}\right) & -r\sin(\theta)\left(\frac{-1-r^2+t^2}{t}\right) & r^2 - \frac{1+r^4-2r^2\cos(2\theta)}{2t^2} - \frac{t^2}{2} \end{pmatrix} \quad (2.6)$$

In this parametrization, the ungauged WZW model action $S(g)$ becomes

$$\begin{aligned} S(g) = \frac{2k}{\pi} \int d^2z \quad \{ & \frac{\bar{\partial}t \partial t (1 - r^2 \cos(2\theta))}{t^2} - \bar{\partial}r \partial r \cos(2\theta) + \bar{\partial}\theta \partial\theta r^2 \cos(2\theta) \\ & + (\bar{\partial}t \partial r + \bar{\partial}r \partial t) \frac{r \cos(2\theta)}{t} + (\bar{\partial}\theta \partial r + \bar{\partial}r \partial\theta) r \sin(2\theta) \\ & - (\bar{\partial}\theta \partial t + \bar{\partial}t \partial\theta) \frac{r^2 \sin(2\theta)}{t} \}. \end{aligned} \quad (2.7)$$

Note that in our gauge the WZ term in $S(g)$ vanishes. The currents (2.4) are given by

$$\begin{aligned}
J_1 &= \left(\frac{r \cos(\theta) + r^3 \cos(3\theta)}{t} - r t \cos(\theta) \right) \partial \theta + \left(r \sin(\theta) - r^3 \sin(3\theta) + \frac{r \sin(\theta)}{t^2} \right) \partial t \\
&+ \left(\frac{\sin(\theta) + r^2 \sin(3\theta)}{t} - t \sin(\theta) \right) \partial r, \\
J_2 &= \left(\frac{r \sin(\theta) - r^3 \sin(3\theta)}{t} - r t \sin(\theta) \right) \partial \theta + \left(-r \cos(\theta) + \frac{r \cos(\theta) - r^3 \cos(3\theta)}{t^2} \right) \partial t \\
&+ \left(\frac{\cos(\theta) + r^2 \cos(3\theta)}{t} + t \cos(\theta) \right) \partial r, \\
J_3 &= -2 r^2 \cos(2\theta) \partial \theta + \frac{2 r^2 \sin(2\theta)}{t} \partial t - 2 r \sin(2\theta) \partial r, \\
J_4 &= \left(\frac{-r \sin(\theta) - r^3 \sin(3\theta)}{t} + r t \sin(\theta) \right) \partial \theta + \left(r \cos(\theta) + \frac{r \cos(\theta) - r^3 \cos(3\theta)}{t^2} \right) \partial t \\
&+ \left(\frac{\cos(\theta) + r^2 \cos(3\theta)}{t} - t \cos(\theta) \right) \partial r, \\
J_5 &= \left(\frac{-r \cos(\theta) - r^3 \cos(3\theta)}{t} - r t \cos(\theta) \right) \partial \theta + \left(r \sin(\theta) + \frac{-r \sin(\theta) + r^3 \sin(3\theta)}{t^2} \right) \partial t \\
&+ \left(\frac{\sin(\theta) - r^2 \sin(3\theta)}{t} - t \sin(\theta) \right) \partial r, \\
J_6 &= 2 r^2 \sin(2\theta) \partial \theta + 2 \left(\frac{r^2 \cos(2\theta) - 1}{t} \right) \partial t - 2 r \cos(2\theta) \partial r,
\end{aligned} \tag{2.8}$$

and similar expressions for \bar{J} 's are obtained from above expressions for J 's with ∂ replaced by $\bar{\partial}$ and a overall minus sign multiplying in J_3, J_6 . Integrating out the gauge fields, we obtain the following result for $S_{gauge}(g)$

$$\begin{aligned}
S_{gauge}(g) &= (2k)/(\pi) \int d^2 z \{ H_{tt} \bar{\partial} t \partial t + H_{rr} \bar{\partial} r \partial r + H_{\theta\theta} \bar{\partial} \theta \partial \theta \\
&+ H_{tr} (\bar{\partial} t \partial r + \bar{\partial} r \partial t) + H_{\theta r} (\bar{\partial} \theta \partial r + \bar{\partial} r \partial \theta) + H_{\theta t} (\bar{\partial} \theta \partial t + \bar{\partial} t \partial \theta) \},
\end{aligned} \tag{2.9}$$

where H 's are defined by

$$H_{ij} = \sum_{k=1}^3 \bar{J}_{k_i} \Delta_{k_j} \quad (i, j = \theta, t, r).$$

Here \bar{J}_{k_i} , Δ_{k_j} are the coefficients of $\bar{\partial} i$ and ∂j in the \bar{J}_k and Δ_k respectively and

$$\begin{aligned}
\Delta_1 &= (1/M) (rt \cos(\theta) \{-1 + 5r^4 + 3r^2 + r^6 + 2r^2 t^2 + 3r^4 t^2 + 3t^2 - 3t^4 + t^6 \\
&+ (r^2 + t^2)(3r^2 t^2 - 8r^2 \cos^2(\theta))\} \partial \theta \\
&+ r \sin(\theta) \{-1 + r^2 + r^4 - r^6 + t^2 - 2r^2 t^2 + r^4 t^2 + t^4 + r^2 t^4 - t^6 - 4r^2 \sin^2(\theta)(1 + r^2 - t^2)\} \partial t \\
&+ t \sin(\theta) \{-1 - r^2 + r^4 + r^6 + 3t^2 - r^2 t^2(r^2 + t^2 - 6 + 8 \sin^2(\theta)) - 3t^4 + t^6\} \partial r), \\
\Delta_2 &= (1/M) (rt \sin(\theta) \{-1 + 5r^4 - 3r^2 - r^6 + 2r^2 t^2 - 3r^4 t^2 - 3t^2 - 3t^4 - t^6 - 8r^2 \sin^2(\theta)(r^2 + t^2)\} \partial \theta
\end{aligned}$$

$$\begin{aligned}
& + r \cos(\theta) \{1 + r^2 - r^4 - r^6 + t^2 + 2r^2 t^2 + r^4 t^2 - t^4 + r^2 t^4 - t^6 - 4r^2 \cos^2(\theta)(1 - r^2 + t^2)\} \partial t \\
& + t \cos(\theta) \{1 - r^2 - r^4 + r^6 + 3t^2 - r^2 t^2 (r^2 + t^2 - 6 + 8 \cos^2(\theta)) + 3t^4 + t^6\} \partial r, \\
\Delta_3 &= \left(\frac{1}{M^{1/2}}\right) (r \{-r^3 - r t^2 + r \cos(2\theta)\} \partial \theta + r \sin(2\theta) \partial r), \\
M &= (1 + r^4 - t^4 - 2r^2 \cos(2\theta))^2.
\end{aligned} \tag{2.10}$$

The resulting effective action describes a three dimensional target space metric that has singularities in $t = 0$ and $M = 0$. As a consequence of the conformal invariance of the theory, the dilaton field must solves the equation $R_{\mu\nu} = \mathcal{D}_\mu \mathcal{D}_\nu \Phi$, and is found to be

$$\Phi = 2 \ln\left(\frac{M}{2t^4}\right) + \alpha, \tag{2.11}$$

where α is a constant. Moreover the antisymmetric tensor is zero.

This background metric and dilaton field were obtained previously in a simple form in reference [3], with a different gauge fixing of the g field (Appendix).

3 VECTOR GAUGING OF $SO(3,1)/E(2)$ WZW MODEL

In this section, we consider the gauging of the WZW model by its Euclidean subgroup $E(2)$. The generators of this subgroup are

$$E_1 = J_1 - K_2, E_2 = J_2 + K_1, E_3 = J_3. \tag{3.1}$$

Obviously in the four dimensional fundamental representation, the two generators E_1, E_2 (generators of translational motions in plane) are null (and also nilpotent)

$$Tr(E_1^2) = Tr(E_2^2) = E_1^3 = E_2^3 = 0. \tag{3.2}$$

As $E(2)$ is a non-abelian subgroup, the only consistent gauged WZW model is the vector gauging [8, 9]. The gauged action is the same as (2.1) with the gauge fields A, \bar{A} taking values in $e(2)$ algebra

$$(A, \bar{A}) = (A_1, \bar{A}_1) E_1 + (A_2, \bar{A}_2) E_2 + (A_3, \bar{A}_3) E_3, \tag{3.3}$$

By using the homomorphism between $SL(2, C)$ and $SO(3, 1)$ groups, we fix the gauge (Appendix) and obtain the same result for g as (2.6). In this parametrization, the ungauged WZW model action $S(g)$ and

the currents (2.4) are like (2.7), (2.8). Integrating out the gauge fields, we obtain the following expression for $S_{gauge}(g)$

$$\begin{aligned}
S_{gauge}(g) &= \frac{2k}{\pi} \int d^2z \left\{ \frac{\bar{\partial}t \partial t r^2 \cos(2\theta)}{t^2} + \bar{\partial}r \partial r \cos(2\theta) - \bar{\partial}\theta \partial\theta r^2 \cos(2\theta) \right. \\
&- (\bar{\partial}t \partial r + \bar{\partial}r \partial t) \frac{r \cos(2\theta)}{t} - (\bar{\partial}\theta \partial r + \bar{\partial}r \partial\theta) r \sin(2\theta) \\
&+ \left. (\bar{\partial}\theta \partial t + \bar{\partial}t \partial\theta) \frac{r^2 \sin(2\theta)}{t} \right\}, \tag{3.4}
\end{aligned}$$

From the above relations (2.7), (3.4) we have

$$S(g, A, \bar{A}) = \frac{2k}{\pi} \int d^2z \left(\frac{\bar{\partial}t \partial t}{t^2} \right), \tag{3.5}$$

which corresponds to the target space metric

$$ds^2 = 2k \frac{dt^2}{t^2} \tag{3.6}$$

and the dilaton field is found to be

$$\Phi = 4 \ln t + \Phi_0, \tag{3.7}$$

where Φ_0 is a constant. The interesting feature of this result is that two degrees of the coset manifold $SO(3,1)/E(2)$ has disappeared, reminiscent of the $SL(2,R)/E(1)$ results [4]. The effective target space action in conformal gauge in terms of $t^2 = e^w$ reads as follows

$$S_{eff} = \frac{k}{2\pi} \int d^2z \partial w \bar{\partial} w + \frac{1}{4\pi} \int d^2z \sqrt{h} R^{(2)} w, \tag{3.8}$$

which is the action of Liouville theory without cosmological constant. As discussed in [6], this dimensional reduction is related to the equivalence of the vector and chiral gauged WZW models when gauged by null subgroups. In fact, in the context of chiral gauging, by using the right subgroup as HG_+ where G_+ is generated by E_1, E_2 , and H by E_3 ; and the left subgroup G_-H where G_- is generated by $E_{-1} = J_1 + K_2$ and $E_{-2} = -J_2 + K_1$, five degrees of freedom of the g field can be fixed and the resultant effective action again becomes one dimensional.

4 VECTOR GAUGING OF $SO(3,1)/SO(3)_b$ WZW MODEL

In the previous section, we noted that the target space obtained by gauging the $SO(3,1)$ WZW model by its null subgroup $E(2)$, degenerated to a one dimensional flat space. We know that the group $SO(3)_b$, obtained

by boosting the generators of $SO(3)$ group, tends to $E(2)$ group in the limit of infinite boost parameter. To understand the mechanism of this two dimensional reduction, we gauge the $SO(3, 1)$ WZW model by its boosted subgroup $SO(3)_b$ and we will see that in the limit of the infinite boost the target space metric and the dilaton field tend to the previous result of the $SO(3, 1)/E(2)$, and for finite but large boost a Toda structure appears. The $SO(3)_b$ group is generated by

$$\begin{aligned}(J_1)_b &= \exp(-\frac{b}{2}K_3) J_1 \exp(\frac{b}{2}K_3) \\ (J_2)_b &= \exp(-\frac{b}{2}K_3) J_2 \exp(\frac{b}{2}K_3) \\ (J_3)_b &= J_3.\end{aligned}\tag{4.1}$$

We fix the gauge freedom and obtain the g field (Appendix) in the form of (2.6) by replacing every t by te^b . The $S(g)$ part of $S(g, A, \bar{A})$ is the same as (2.7). The currents (2.4) are the same as (2.8) with the difference that every t must be replaced by te^b . The gauge fields are

$$(A, \bar{A}) = (A_1, \bar{A}_1) (J_1)_b + (A_2, \bar{A}_2) (J_2)_b + (A_3, \bar{A}_3) J_3.\tag{4.2}$$

Integrating them out, we obtain

$$\begin{aligned}S_{gauge}(g) &= \frac{2k}{\pi} \int d^2z \left[\frac{\bar{\partial}t \partial t r^2 \cos(2\theta)}{t^2} + \bar{\partial}r \partial r (\cos(2\theta) + \frac{4e^{-2b}}{t^2}) \right. \\ &- \bar{\partial}\theta \partial\theta (r^2 \cos(2\theta) + \frac{4r^2 e^{-2b}}{t^2}) - (\bar{\partial}t \partial r + \bar{\partial}r \partial t) (\frac{r \cos(2\theta)}{t} - \frac{2re^{-2b}}{t^3}) \\ &- \left. (\bar{\partial}\theta \partial r + \bar{\partial}r \partial\theta) r \sin(2\theta) + (\bar{\partial}\theta \partial t + \bar{\partial}t \partial\theta) \frac{r^2 \sin(2\theta)}{t} + O(e^{-4b}) \right].\end{aligned}\tag{4.3}$$

Then the action of $SO(3, 1)/SO(3)_b$ reads

$$S(g, A, \bar{A}) = \frac{2k}{\pi} \int d^2z \left[\frac{\bar{\partial}t \partial t}{t^2} + e^{-2b} \left(\frac{4\bar{\partial}r \partial r}{t^2} - \frac{2r(\bar{\partial}t \partial r + \bar{\partial}r \partial t)}{t^3} + 4r^2 \frac{\bar{\partial}\theta \partial\theta}{t^2} \right) + O(e^{-4b}) \right],\tag{4.4}$$

and the dilaton field is found to be

$$\Phi = 4 \ln t + \Phi_0 + O(e^{-4b}).\tag{4.5}$$

To diagonalize (4.4), we define

$$r^2 = e^V, t = (e^W + 2e^{-2b+V})^{1/2}, \theta = X/2,\tag{4.6}$$

and get

$$S_{eff} = \frac{k}{2\pi} \int d^2z \{ \bar{\partial}W \partial W + 4\epsilon^2 e^{-W+V} (-\bar{\partial}W \partial W + \bar{\partial}X \partial X + \bar{\partial}V \partial V) + O(\epsilon^4) \}$$

$$+ \frac{1}{4\pi} \int d^2 z \sqrt{h} R_{(2)} \{W + 2\epsilon^2 e^{-W+V} + O(\epsilon^4)\}, \quad (4.7)$$

where $\epsilon = e^{-b}$. We note that in the infinite boost limit, $\epsilon \rightarrow 0$, the above effective action tends to the previous result for $SO(3,1)/E(2)$ model. For small ϵ , the effective action is the effective action of Toda field W that interacts with the matter field X, V . Elliminating X field by its equation of motion from (4.7), we find

$$\begin{aligned} S_{eff} &= \frac{k}{2\pi} \int d^2 z \{ \bar{\partial} W \partial W + 4\epsilon^2 f e^{W-V} + 4\epsilon^2 e^{V-W} (-\bar{\partial} W \partial W + \bar{\partial} V \partial V) + O(\epsilon^4) \} \\ &+ \frac{1}{4\pi} \int d^2 z \sqrt{h} R_{(2)} \{W + 2\epsilon^2 e^{-W+V} + O(\epsilon^4)\}, \end{aligned} \quad (4.8)$$

where f is an arbitrary constant. The first term shows the Toda structure for W field, and the absence of kinetic term for the other Toda field, the V field, is the result of the existence of just one Cartan generator J_3 in the gauged ubalgebra. The second term in (4.8) is in the form of a potential term in Toda theory. The other terms describe the interaction of the Toda fields W and V . Here we have a transition at $b = 0$ from the black hole metric given by (2.7), (2.9) to a Toda structure at $b \rightarrow \infty$, and between these two limits, the effects of black hole structure appear in the form of interaction terms of the Toda fields.

5 VECTOR GAUGING OF $SO(3,1)$ WZW MODEL BY ITS GENERALIZED NON-STANDARD BILINEAR FORM

It is known that the bilinear form that is used in construction of WZW models, must be non-degenerate and several interesting models with a bilinear form different from the Killing form have been constructed [10].

In this article we have used the usual bilinear form of Lorentz group i.e. its Killing form

$$\Omega = 2 \left\{ \sum_{i=1}^3 (-e_{ii} + e_{(i+3)(i+3)}) \right\}. \quad (5.1)$$

However, there is also a second non-degenerate bilinear form that can be used for $SO(3,1)$, i.e.

$$\Omega' = \sum_{i=1}^3 \{ e_{i(i+3)} + e_{(i+3)i} \}. \quad (5.2)$$

We can use a linear combination of the two bilinear forms (5.1) and (5.2)

$$\Omega(\lambda) = \frac{\Omega}{2} + \lambda \Omega', \quad (5.3)$$

in our construction.

The ungauged WZW action is

$$S(g) = \frac{k}{4\pi} \int d^2 z J_i \Omega_{ij}(\lambda) \tilde{J}_j, \quad (5.4)$$

where \tilde{J}_i are obtained from (2.8) by $\partial \rightarrow \bar{\partial}$. The gauged part of (2.1), is

$$S_{gauge}(g) = \frac{k}{2\pi} \int d^2 z \{A_i \bar{J}_j - \bar{A}_i J_j + A_i \bar{A}_j - (g^{-1} A g)_i \bar{A}_j\} \Omega_{ij}(\lambda). \quad (5.5)$$

After integrating out the gauge fields, the target space metric for $SO(3)$ subgroup becomes

$$ds^2 = \frac{k}{\pi} G_{ij}(\lambda) dx^i dx^j, \quad x^i = t, \theta, \phi \quad (5.6)$$

where

$$\begin{aligned} G_{tt}(\lambda) &= \frac{2}{1+\lambda^2} + 2 \frac{\lambda^2}{1+\lambda^2} \left[\frac{\lambda^2}{\cos^2 \phi \sin^2 \theta} - 4 \frac{\lambda \tan \phi \cot \theta}{\sinh 2t \sin \theta} \right. \\ &\quad \left. + \frac{\cosh^4 t - \cos^2 \phi \cosh^2 t - \cos^2 \theta \cos^2 \phi \sinh^2 t + \sinh^2 t \cos^2 \phi \sin^2 \theta \cosh^2 t}{\sinh^2 t \cosh^2 t \sin^2 \theta \cos^2 \phi} \right], \\ G_{\theta\theta} &= \frac{2}{1+\lambda^2} \{\coth^2 t + \lambda^2\}, \\ G_{\phi\phi} &= \frac{2}{1+\lambda^2} \left\{ \frac{\tanh^2 t}{\sin^2 \theta} + \cot^2 \theta \coth^2 t \tan^2 \phi - 4 \frac{\lambda \tan \phi \cot \theta}{\sinh 2t \cos \theta} + \lambda^2 \left(1 + \frac{\cot^2 \theta}{\cos^2 \phi}\right) \right\}, \\ G_{t\theta} &= G_{\theta t} = \frac{2\lambda}{1+\lambda^2} \left\{ -\frac{\lambda^2 \tan \phi}{\sin \theta} + 2 \frac{\lambda \cot \theta}{\sinh 2t} - \coth^2 t \frac{\tan \phi}{\sin \theta} \right\}, \\ G_{t\phi} &= G_{\phi t} = \frac{2\lambda}{1+\lambda^2} \left\{ \lambda^2 \cos \theta (\sin^2 \theta \cosh^2 t - \frac{1}{\sin^2 \theta \cos^2 \phi}) + 2\lambda \tan \phi \frac{1 + \cos^2 \theta}{\sin^2 \theta \sinh 2t} \right. \\ &\quad \left. + 4 \cos \theta \left(\frac{-\cosh^4 t (1 + \cos^2 \phi \sin^2 \theta) + \cos^2 \phi (-1 + \sin^2 \theta \cosh^2 t + 2 \cosh^2 t)}{\sinh^2 2t \cos^2 \phi \sin^2 \theta} \right) \right. \\ &\quad \left. + \frac{1 + \cosh^2 t \sin^2 \theta}{2(1 + \lambda^2)} \right\}, \\ G_{\theta\phi} &= G_{\phi\theta} = \frac{2}{1+\lambda^2} \left\{ \lambda^2 \cot \theta \tan \phi - \frac{2\lambda}{\sin \theta \sinh 2t} + \cot \theta \tan \phi \cosh^2 t \right\}, \end{aligned} \quad (5.7)$$

and the dilaton field becomes

$$\Phi = \ln \{2 \sin^2 \theta \sinh^2 2t \cos^2 \phi (1 + \lambda^2)\}. \quad (5.8)$$

The central charge of coset model is

$$c = 6 \frac{k(k-4) + \lambda^2}{(k-4)^2 + \lambda^2} - \frac{3k}{k+2}, \quad (5.9)$$

that by the following redefinition of k ,

$$k' = \frac{D+4M}{M-D} \pm \frac{1}{M-D} \sqrt{9D^2 + 16M^2} \quad (5.10)$$

where

$$\begin{aligned} D &= k^3 + 4k^2 + (\lambda^2 - 32)k + 4\lambda^2, \\ M &= k^3 - 6k^2 + \lambda^2 k + 2\lambda^2 + 32 \end{aligned} \tag{5.11}$$

could be written as

$$c = 3 \frac{k'^2 + 8k'}{(k' + 2)(k' - 4)}, \tag{5.12}$$

that is the usual central charge of $SO(3,1)/SO(3)$ coset model.

If we set $\lambda = 0$ we recover the target space metric and the dilaton field that obtained in reference [3]. In addition, we find that the singularity structure of (5.6) is like the case with $\lambda = 0$; therefore the singularity structure doesn't change when using the generalized bilinear form in place of the usual Killing form of the Lorentz group.

In the case of gauging the $E(2)$ subgroup, the resultant effective action is in the form (3.8) with an overall numerical factor $1 + \lambda^2$. This simple renormalization of the action, has been independently noted in a study of the non-standard $SO(3,1)$ WZW model [11].

6 CONCLUSIONS

By studying the structure of the vector G/H WZW model, when G is the Lorentz group and H the non-semisimple null Euclidean subgroup in two dimensions $E(2)$, we noticed that only one dimension of $SO(3,1)/E(2)$ three dimensional coset manifold survives in the target space metric and the dilaton field of the gauged theory. To see the disappearance of the two dimensions from the target space and dilaton field, we studied the $SO(3,1)/SO(3)$ vector gauged model that has a rich singularity structure in its target metric. By boosting the $SO(3)$ gauged subgroup, at the limit of infinite boost, the singularity structure disappears and the three dimensional target space metric reduces to our one dimensional flat target space.

This dimensional reduction, on the other hand can be related to the equivalence of vector and chiral gauged Lorentz WZW models by appropriate null subgroups [6].

It must be noticed that in [7], the gauging of maximally non-compact groups based on the Gauss decomposition, by some of their null subgroups was considered. In particular the null gauging of non-compact $SO(3,1)$

has also considered; however, their gauging is different from what we consider here. They considered the gauging of $SO(3, 1)$ by one left and one right null subgroup which are generated by a pair of null generators and the resultant target space metric is different from the one considered here for vector gauging of $E(2)$ subgroup. In particular, the background metric is four dimensional and no dimensional reduction occurs.

All the above considerations can be applied to the $SO(n, 1)/E(n - 1)$ vector WZW models, in which cases we also obtain a one dimensional target space and a dilaton field that can be interpreted as a Liouville field with a background charge. The $SO(n, 1)/SO(n)_b$ vector WZW models also tend to the corresponding $SO(n, 1)/E(n - 1)$ models in the limit of infinite boost; and in the finite but large limit of boost parameter, the interactions of Liouville field with the other remaining degrees of freedom occur, giving Toda like theories.

7 APPENDIX

For every element of $SL(2, C)$ group in the parametrization form $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, there is a 4×4 matrix in the fundamental representation of $SO(3, 1)$ given by [12],

$$\begin{pmatrix} \frac{1}{2}(\alpha\alpha^* + \beta\beta^* + \gamma\gamma^* + \delta\delta^*) & \Re(\alpha\beta^* + \gamma\delta^*) & -\Im(\alpha\beta^* + \gamma\delta^*) & \frac{1}{2}(\alpha\alpha^* - \beta\beta^* + \gamma\gamma^* - \delta\delta^*) \\ \Re(\alpha\gamma^* + \beta\delta^*) & \Re(\alpha\delta^* + \beta\gamma^*) & -\Im(\alpha\delta^* - \beta\gamma^*) & \Re(\alpha\gamma^* - \beta\delta^*) \\ \Im(\alpha\gamma^* + \beta\delta^*) & \Im(\alpha\delta^* + \beta\gamma^*) & \Re(\alpha\delta^* - \beta\gamma^*) & \Im(\alpha\gamma^* - \beta\delta^*) \\ \frac{1}{2}(\alpha\alpha^* + \beta\beta^* - \gamma\gamma^* - \delta\delta^*) & \Re(\alpha\beta^* - \gamma\delta^*) & -\Im(\alpha\beta^* - \gamma\delta^*) & \frac{1}{2}(\alpha\alpha^* - \beta\beta^* - \gamma\gamma^* + \delta\delta^*) \end{pmatrix} \quad (7.1)$$

Let us take the parametrization $\begin{pmatrix} e^{i\lambda} & 0 \\ \nu & e^{-i\lambda} \end{pmatrix}$; $0 \leq \lambda \leq 2\pi, \nu \in C$ for the group elements of $E(2)$ [13], then

by using the vector gauge freedom, every element of $SL(2, C)/E(2)$ can be written as $\begin{pmatrix} re^{i\theta} & t \\ \frac{-1+r^2e^{2i\theta}}{t} & re^{i\theta} \end{pmatrix}$ and by using the above relation (7.1) we obtain the equation (2.6) for gauged fixed g field. Similarly for

the $SO(3)$ case, we obtain also (2.6). The gauged fixed field g that was used in reference [3], written in our $SL(2, C)$ language, is $\begin{pmatrix} A + iB & iC \\ iC & \frac{1-C^2}{A+iB} \end{pmatrix}$, where A, B, C are real fields.

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